MATH2118 Lecture Notes

Further Engineering Mathematics C

Infinite Series

Dr Eddie Ly

School of Mathematical & Geospatial Sciences (SMGS)

College of Sciences, Engineering & Health (SEH)

RMIT University, Melbourne, Australia

Summer Course 2015

Contents

1 Introduction **3**

2 Convergent Series **3**

2.1 Geometric Series 6

2.2 A Test for Divergent 7

3 Positive Term Series **8**

3.1 Comparison Test 9

3.2 Limit Comparison Test 11

3.3 Ratio Test 13

4 Alternating Series **15**

4.1 Alternating Series Test 15

4.2 Error in Approximating the Sum of an Alternating Series 17

4.3 Absolute Convergence 17

4.4 Conditionally Convergent Theorem 19

5 Review Questions **20**

6 Answers to Review Questions **23**

1 Introduction

If $\{a_n\}$ is the sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$, then the sum

$$\boxed{\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + a_{n+1} + \cdots}$$

is called an *infinite series*, or simply a *series*. Often, we can write $\sum_{n=1}^{\infty} a_n$ as $\sum a_n$). The a_n, $n = 1, 2, 3, \ldots$, are called the *terms* of the series; a_n is called the *general term*. Associated with every infinite series $\sum a_n$, there is a *sequence of partial sums* $\{S_n\}$ whose terms are defined by

$$\begin{aligned}
S_1 &= a_1, \\
S_2 &= \underbrace{a_1}_{=S_1} + a_2 = S_1 + a_2, \\
S_3 &= \underbrace{a_1 + a_2}_{=S_2} + a_3 = S_2 + a_3, \\
S_n &= \sum_{m=1}^{n} a_m \\
&= \underbrace{a_1 + a_2 + a_3 + \cdots + a_{n-1}}_{=S_{n-1}} + a_n \\
&= S_{n-1} + a_n, \quad \text{and so on.}
\end{aligned}$$

The term $S_n = \sum_{m=1}^{n} a_m = a_1 + a_2 + \cdots + a_n$ of this sequence is called the *nth partial sum* of the series.

2 Convergent Series

An infinite series $\sum_{n=1}^{\infty} a_n$ is said to be *convergent* if the sequence of partial sums $\{S_n\}$ converges. That is,

$$\boxed{\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{m=1}^{n} a_m = L.}$$

The number L is the *sum of the series*. If $\lim\limits_{n\to\infty} S_n = L$ *does not exist*, the series is said to be *divergent*.

EXAMPLE

Show that the series $\sum \frac{1}{(n+4)(n+5)}$ is convergent.

SOLUTION

Considering the general term of the series:

$$\begin{aligned} a_n &= \frac{1}{(n+4)(n+5)} \\ &= \frac{1}{n+4} - \frac{1}{n+5}. \end{aligned}$$

The nth partial sum of the series is

$$\begin{aligned} S_n &= \left[\frac{1}{5} - \underbrace{\frac{1}{6}\right] + \left[\frac{1}{6}} - \underbrace{\frac{1}{7}\right] + \left[\frac{1}{7}} - \underbrace{\frac{1}{8}\right] + \cdots} + \underbrace{\left[\frac{1}{n+4}\right]} - \frac{1}{n+5} \\ &= \frac{1}{5} - \frac{1}{n+5}. \end{aligned}$$

Since

$$\begin{aligned} \lim_{n\to\infty} S_n &= \lim_{n\to\infty} \left(\frac{1}{5} - \frac{1}{n+5}\right) \\ &= 1/5 \\ &= L, \end{aligned}$$

the series converges, and we write

$$\sum \frac{1}{(n+4)(n+5)} = \frac{1}{5}.$$

REMARK:

Because of the manner in which the general term of this sequence of partial sums "collapses" to two terms, this series known as a *telescoping series*.

EXAMPLE

Show that the series $\sum \ln\big(1 + 1/n\big)$ is divergent.

SOLUTION

Partial sum:

$$\begin{aligned} S_n &= \sum_{m=1}^{n} \ln\left(1 + \frac{1}{m}\right) \\ &= \sum_{m=1}^{n} \ln\left(\frac{m+1}{m}\right) \\ &= \sum_{m=1}^{n} \Big(\ln(m+1) - \ln m\Big) \end{aligned}$$

$$
\begin{aligned}
&= (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + (\ln 4 - \ln 3) + \cdots + \big(\ln(n) - \ln(n-1)\big) \\
&\qquad + \big(\ln(n+1) - \ln n\big) \\
&= -\ln 1 + \ln(n+1) \\
&= \ln(n+1) \quad \text{since } \ln 1 = 0.
\end{aligned}
$$

As $n \to \infty$, $n+1 \to \infty$, so $S_n = \ln(n+1) \to \infty$. Hence, $\lim_{n\to\infty} S_n = L$ does not exist, and the series diverges.

EXAMPLE

If S_n denotes the nth partial sum of $\sum \frac{3}{(3n-2)(3n+1)}$, show that

$$S_n = 1 - \frac{1}{3n+1}.$$

Deduce that the series converges. What is its sum ?

SOLUTION

Partial sum:

$$
\begin{aligned}
S_n &= \sum_{m=1}^{n} \frac{3}{(3m-2)(3m+1)} \\
&= \sum_{m=1}^{n} \left(\frac{1}{3m-2} - \frac{1}{3m+1} \right) \\
&= \left(\frac{1}{1} - \underbrace{\frac{1}{4}\right) + \left(\frac{1}{4}} - \underbrace{\frac{1}{7}\right) + \left(\frac{1}{7}} - \underbrace{\frac{1}{10}\right) + \left(\frac{1}{10}} - \underbrace{\frac{1}{13}\right) + \cdots} \\
&\qquad \underbrace{+\left(\frac{1}{3n-5}} - \underbrace{\frac{1}{3n-2}\right) + \left(\frac{1}{3n-2}} - \frac{1}{3n+1}\right) \\
&= 1 - \frac{1}{3n+1}.
\end{aligned}
$$

As $n \to \infty$, $S_n \to 1$ since $\frac{1}{3n+1} \to 0$. Hence, the series converges to the sum of

$$L = \lim_{n\to\infty} S_n = 1.$$

2.1 Geometric Series

A *geometric series* is a series having the form

$$\boxed{\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n + \cdots}$$

and its sum,

$$\begin{aligned} S_n &= \sum_{m=0}^{n-1} ar^m \\ &= a + ar + ar^2 + + \cdots + ar^{n-1} \\ &= a\left(\frac{1-r^n}{1-r}\right) \quad \text{if } r \neq 1, \end{aligned}$$

where r is the *common ratio*. The series converges for $|r| < 1$ since $\lim_{n\to\infty} S_n = \dfrac{a}{1-r}$. Note that $r^n \to 0$ as $n \to \infty$ for $|r| < 1$, and diverges for $|r| \geq 1$.

EXAMPLE

Determine the nth partial sum of $\sum (3/4)^{n+1}$, and determine if the series converges.

SOLUTION

Series:

$$\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^{n+1} = \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \left(\frac{3}{4}\right)^4 + \cdots + \left(\frac{3}{4}\right)^n + \cdots .$$

This is a geometric series with $a = 3/4$ and $r = 3/4 < 1$. Thus,

$$\begin{aligned} S_n &= \frac{3}{4}\left(\frac{1-(3/4)^n}{1-3/4}\right) \\ &= 3\big(1-(3/4)^n\big). \end{aligned}$$

As $n \to \infty$, $(3/4)^n \to 0$ (since $3^n < 4^n$), so

$$\lim_{n\to\infty} S_n = L = 3.$$

The series converges with the sum of 3.

2.2 A Test for Divergent

If a_n is the general terms of a series, and S_n is the corresponding sequence of partial sums, then $a_n = S_n - S_{n-1}$ (since $S_n = S_{n-1} + a_n$).

Now, if the series converges to a number L, we have $\lim_{n\to\infty} S_n = L$, and $\lim_{n\to\infty} S_{n-1} = L$. This implies that

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \big(S_n - S_{n-1}\big)\\ &= \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}\\ &= L - L\\ &= 0.\end{aligned}$$

That is,

$$\textit{if} \quad \sum_{n=1}^{\infty} a_n \ \textit{converges}, \quad \text{then} \ \lim_{n\to\infty} a_n = 0.$$

On the other hand, if $\lim_{n\to\infty} a_n = 0$, we cannot deduce that the series $\sum a_n$ converges. We can at least say that if $\lim_{n\to\infty} a_n \neq 0$, then the *series must diverge*.

Test for a divergence series:

$$\boxed{\text{If } \lim_{n\to\infty} a_n \neq 0 \text{ then } \sum_{n=1}^{\infty} a_n \text{ diverges.}}$$

EXAMPLE

Consider the series $\sum \dfrac{n}{n-1}$. Rewriting the general term of the series as

$$\begin{aligned}a_n &= \frac{n}{n-1}\\ &= \frac{(n-1)+1}{n-1}\\ &= 1 + \frac{1}{n-1}.\end{aligned}$$

Since

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty}\left(1 + \frac{1}{n-1}\right)\\ &= 1\\ &\neq 0,\end{aligned}$$

the series diverges by the nth term test for divergence.

▌EXAMPLE

Consider the infinite series $\sum \frac{4n-1}{5n+3}$.

Here $a_n = \frac{4n-1}{5n+3}$. It follows from the test for a divergent series that this series must diverge, since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{4n-1}{5n+3} = \lim_{n\to\infty} \frac{4-1/n}{5+3/n} = 4/5 \neq 0.$$

▌EXAMPLE

Consider the p-series, $\sum \frac{1}{n^4}$. Since

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{1}{n^4} \\ &= 0,\end{aligned}$$

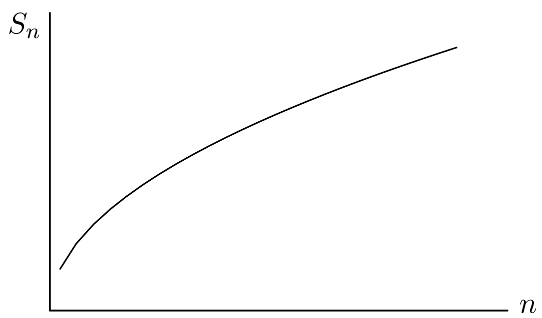
we cannot conclude whether this series converges or diverges by the nth term test for divergence.

3 Positive Term Series

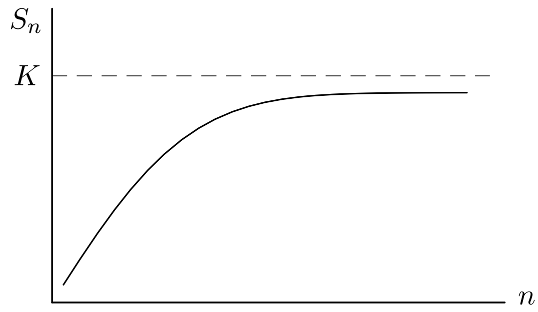
A series of the form $\sum_{n=1}^{\infty} a_n$, where every $a_n > 0$, is called *positive term series.* The sequence of partial sums, $S_n = \sum_{m=1}^{n} a_m$ (all $a_m > 0$), is *monotonically increasing.* That is, $S_{n-1} < S_n < S_{n+1}$ for all n.

There are only two possibilities to consider:

(1) S_n increases without bound. That is, $S_n \to \infty$ as $n \to \infty$, and the series must diverge.



(2) The partial sums are bounded above by some constant K, such that $S_n < K$ for all n. Hence, S_n must approach to some limit ($\leq K$), and the series must converge (*monotonic convergence theorem*).



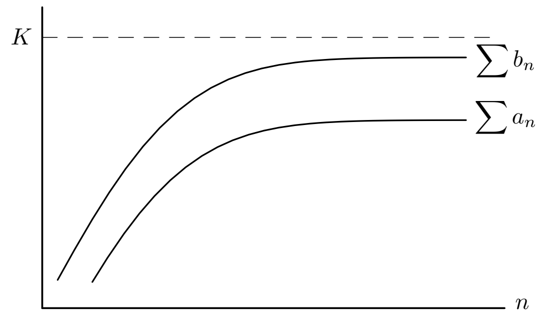
3.1 Comparison Test

It is often possible to determine convergence or divergence of a series $\sum a_n$ by comparing its terms with the terms of a "test series" $\sum b_n$ that is known to be convergent or divergent.

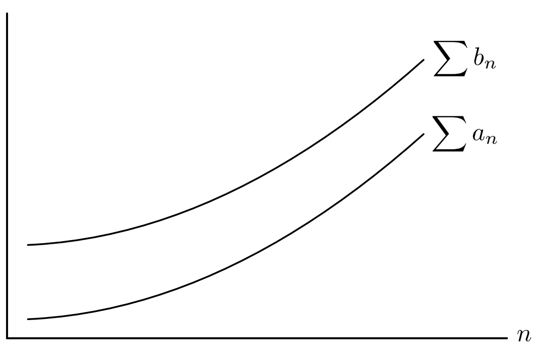
Suppose $\sum a_n$ (represents a "smaller series") and $\sum b_n$ ("larger series") are two positive term series, such that

$$0 \leq a_n \leq b_n \quad \text{for all } n \text{ (or at least for sufficiently large } n.)$$

- If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
  (If the larger series converges, the smaller series must converge).



- If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.
  (If the smaller series diverges, the larger series must diverge).



Common standard series used for comparison:

- *Geometric series:*
  $\sum ar^n$ converges to $\dfrac{a}{1-r}$ if $|r| < 1$, and diverges if $|r| \geq 1$.
- *p-series:*
  $$\sum \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$
  converges if $p > 1$, and diverges if $p \leq 1$.
  If $p = 1$, the divergent series $\sum \dfrac{1}{n}$ is called *harmonic series.*

EXAMPLE

Test for convergence of $\sum \dfrac{n}{n^3+4}$.

SOLUTION

We observe that

$$\frac{n}{n^3+4} < \frac{n}{n^3} = \frac{1}{n^2} \quad \text{since } n^3+4 > n^3 \text{ for all } n.$$

Because the larger series $\sum \dfrac{1}{n^2}$ (p-series, $p = 2 > 1$) converges, then the smaller series $\sum \dfrac{n}{n^3+4}$ must converge by the comparison test.

EXAMPLE

Test for convergence of $\sum \dfrac{\ln(n+2)}{n}$.

SOLUTION

Since $\ln(n+2) > 1$ for $n \geq 1$, we have

$$\frac{\ln(n+2)}{n} > \frac{1}{n}.$$

Since the smaller series $\sum \dfrac{1}{n}$ (p-series, $n = 1$) is a divergent harmonic series, then by the comparison test the larger series $\sum \dfrac{\ln(n+2)}{n}$ must diverge.

▌EXAMPLE

Consider the series $\sum \dfrac{100+\cos(5n)}{n^3}$.

SOLUTION

Since $\cos(5n) \leq 1$, we have

$$\begin{aligned} 100+\cos(5n) &\leq 101 \\ \Rightarrow \quad \frac{100+\cos(5n)}{n^3} &\leq \frac{101}{n^3} \end{aligned}$$

Since the larger series

$$\sum \frac{101}{n^3} = 101 \sum \frac{1}{n^3},$$

which is a p-series with $p = 3 > 1$, is a convergent series, then $\sum \dfrac{100+\cos(5n)}{n^3}$ also converges.

3.2 Limit Comparison Test

Suppose $\sum a_n$ and $\sum b_n$ are positive term series, and that

$$\boxed{\lim_{n\to\infty} \frac{a_n}{b_n} = c.}$$

Then,

- if $c > 0$, then both series either convergent or divergent.
- if $c = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
- if $c = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

The limit comparison test is often applicable to series for which the comparison test is inconvenient.

▌EXAMPLE

Test for convergence of $\sum \dfrac{n}{(8n^5+7)^{1/3}}$.

SOLUTION

For large n values, the general term of this series $a_n = \dfrac{n}{(8n^5+7)^{1/3}}$ "behaves" like

$$\frac{n}{(n^5)^{1/3}} = \frac{n}{n^{5/3}} = \frac{1}{n^{2/3}}.$$

Let $\sum b_n = \sum \frac{1}{n^{2/3}}$ be the test series, we have

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty}\left(\frac{n}{(8n^5+7)^{1/3}} \times \frac{n^{2/3}}{1}\right) \\ &= \lim_{n\to\infty} \frac{n^{5/3}}{(8n^5+7)^{1/3}} \\ &= \lim_{n\to\infty}\left(\frac{n^5}{8n^5+7}\right)^{1/3} \\ &= \lim_{n\to\infty}\left(\frac{1}{8+\frac{7}{n^5}}\right)^{1/3} \\ &= \big(1/8\big)^{1/3} \\ &= 1/2.\end{aligned}$$

Here, $c = 1/2 > 0$, and both series either converge or diverge. Since $\sum b_n = \sum \frac{1}{n^{2/3}}$ is a divergent p-series ($p = 2/3 < 1$), then $\sum \frac{n}{(8n^5+7)^{1/3}}$ must diverge by the limit comparison test.

▌**EXAMPLE**

Test for convergence of $\sum \frac{n^3+5n-\sqrt{n}+2}{n^5+5}$.

SOLUTION

For large n values, the general term of this series behaves like

$$\frac{n^3+5n-\sqrt{n}+2}{n^5+5} \sim \frac{n^3}{n^5} = \frac{1}{n^2}.$$

Let $\sum b_n = \sum \frac{1}{n^2}$ be the test series (convergent p-series), we have

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty}\left(\frac{n^3+5n-\sqrt{n}+2}{n^5+5} \times \frac{n^2}{1}\right) \\ &= \lim_{n\to\infty} \frac{n^5+5n^3-n^{5/2}+2n^2}{n^5+5} \\ &= \lim_{n\to\infty} \frac{1+\frac{5}{n^2}-\frac{1}{n^{5/2}}+\frac{2}{n^3}}{1+\frac{5}{n^5}} \\ &= 1.\end{aligned}$$

Here $c = 1 > 0$, and both series converge or diverge. Since $\sum b_n = \sum \frac{1}{n^2}$ converges, then the series $\sum \frac{n^3+5n-\sqrt{n}+2}{n^5+5}$ also converges by the limit comparison test.

3.3 Ratio Test

Ratio test is useful when a_n involves factorials and nth powers of a constant or n.

Suppose $\sum a_n$ is a positive term series, such that

$$\boxed{\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L.}$$

Then,

- if $L < 1$, the series converges;
- if $L > 1$, the series diverges;
- if $L = 1$, the ratio test is inconclusive (no indication of whether the series converges or diverges).

EXAMPLE

Test for convergence of $\sum \frac{5^n}{n!}$.

SOLUTION

Here $a_n = \frac{5^n}{n!}$, then

$$\begin{aligned} a_{n+1} &= \frac{5^{n+1}}{(n+1)!} \\ &= \frac{5^n\,5}{(n+1)\,n!}, \\ \Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \left(\frac{5^n\,5}{(n+1)\,n!} \times \frac{n!}{5^n} \right) \\ &= \lim_{n\to\infty} \frac{5}{n+1} \\ &= 0 \\ &= L. \end{aligned}$$

Since $L = 0 < 1$, this series must converge by the ratio test.

EXAMPLE

Test for convergent of $\sum \frac{n^n}{n!}$.

SOLUTION

Here $a_n = \frac{n^n}{n!}$. Then,

$$
\begin{aligned}
a_{n+1} &= \frac{(n+1)^{n+1}}{(n+1)!} \\
&= \frac{(n+1)(n+1)^n}{(n+1)\,n!} \\
&= \frac{(n+1)^n}{n!}, \\
\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \left(\frac{(n+1)^n}{n!} \times \frac{n!}{n^n} \right) \\
&= \lim_{n\to\infty} \left(\frac{n+1}{n} \right)^n \\
&= \lim_{n\to\infty} \big(1 + 1/n\big)^n \\
&= e.
\end{aligned}
$$

Since $L = e > 1$, this series must diverge by the ratio test.

EXAMPLE

Test for convergence of $\sum \frac{2^n}{(2n)!}$.

SOLUTION

Here $a_n = \frac{2^n}{(2n)!}$. Then,

$$
\begin{aligned}
a_{n+1} &= \frac{2^{n+1}}{(2n+2)!} \\
&= \frac{2^{n+1}}{(2n+2)(2n+1)(2n)!} \\
&= \frac{2^n}{(n+1)(2n+1)(2n)!}, \\
\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \left(\frac{2^n}{(n+1)(2n+1)(2n)!} \times \frac{(2n)!}{2^n} \right) \\
&= \lim_{n\to\infty} \frac{1}{(n+1)(2n+1)} \\
&= 0.
\end{aligned}
$$

Since $L = 0 < 1$, this series converges by the ratio test.

EXAMPLE

The ratio test will give inconclusive answer when applied to a p-series $\sum \frac{1}{n^p}$:

$$\begin{aligned}\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{n^p}{(n+1)^p} \\ &= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^p \\ &= \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right)^p \\ &= 1^p \\ &= 1 = L \quad \text{for all } p \text{ values.}\end{aligned}$$

4 Alternating Series

A series having either form,

$$a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{n+1} a_n + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} a_n,$$

$$-a_1 + a_2 - a_3 + a_4 - \cdots + (-1)^n a_n + \cdots = \sum_{n=1}^{\infty} (-1)^n a_n,$$

where $a_n > 0$ for $n = 1, 2, 3, \ldots$, and the terms are alternately positive and negative, is said to be an *alternating series*. For example,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots ;$$

$$\sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{2^n} = \frac{\ln 2}{4} - \frac{\ln 3}{8} + \frac{\ln 4}{16} - \frac{\ln 5}{32} + \cdots ;$$

$$\sum_{n=0}^{\infty} \frac{\cos(n\pi)}{n^2+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1} = 1 - \frac{1}{2} + \frac{1}{5} - \frac{1}{10} + \frac{1}{17} + \cdots .$$

4.1 Alternating Series Test

Consider the series $\sum (-1)^{n+1} a_n$, where each $a_n > 0$.

If $\lim_{n\to\infty} a_n = 0$ and $a_{n+1} < a_n$ for all n, then the series converges.

EXAMPLE

Show that the alternating harmonic series $\sum \frac{(-1)^{n+1}}{n}$ converges.

SOLUTION

Here $a_n = \frac{1}{n}$, and $a_{n+1} = \frac{1}{n+1}$. We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0,$$

and

$$\frac{1}{n+1} < \frac{1}{n} \quad \Rightarrow a_{n+1} < a_n \text{ for all } n.$$

Since $\lim_{n\to\infty} a_n = 0$ and $a_{n+1} < a_n$ for all n (a_n is monotonically decreasing to zero), it follows that the alternating harmonic series converges.

EXAMPLE

Test for convergence of $\sum (-1)^{n+1} \frac{\sqrt{n}}{n+1}$.

SOLUTION

Here $a_n = \frac{\sqrt{n}}{n+1}$ and $a_{n+1} = \frac{\sqrt{n+1}}{n+2}$. To show that the terms of the series satisfy the condition $a_{n+1} < a_n$, we let $f(n) = a_n = \frac{\sqrt{n}}{n+1}$. From

$$\begin{aligned} f'(n) = \frac{df}{dn} &= \frac{d}{dn}\left(\frac{\sqrt{n}}{n+1}\right) \\ &= \frac{\frac{1}{2\sqrt{n}}(n+1) - \sqrt{n}\,(1)}{(n+1)^2} \\ &= \frac{\frac{1}{2}(n+1) - n}{\sqrt{n}(n+1)^2} \\ &= \frac{\frac{1}{2}(1-n)}{\sqrt{n}(n+1)^2} \\ &= \frac{-(n-1)}{2\sqrt{n}(n+1)^2}, \end{aligned}$$

we can see that $f'(n) < 0$ for $n > 1$. That is, function $f(n)$ decreases for $n > 1$. Thus, $a_{n+1} < a_n$ is true for $n > 1$. Also,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sqrt{n}}{n+1} = 0.$$

Hence, this series converges by the alternating series test.

EXAMPLE

Consider $\sum(-1)^{n+1}\dfrac{2n+1}{3n-1}$.

SOLUTION

Since

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty}\frac{2n+1}{3n-1}\\ &= \lim_{n\to\infty}\frac{2+\frac{1}{n}}{3-\frac{1}{n}}\\ &= 2/3\\ &\neq 0,\end{aligned}$$

the series diverges by the alternating series test.

4.2 Error in Approximating the Sum of an Alternating Series

Suppose the alternating series $\sum(-1)^{n+1}a_n$, where $a_n > 0$, converges to a number L. If S_n is the nth partial sum of the series, and $a_{n+1} < a_n$ for all n values, then

$$\boxed{\big|L - S_n\big| \leq a_{n+1} \quad \text{for all } n.}$$

The *error* of the series is less than the absolute value of the $(n+1)$th term of the series.

4.3 Absolute Convergence

If $\sum a_n$ is *absolutely convergent*, that is, if $\sum|a_n|$ converges, then $\sum a_n$ converges.

However, if $\sum|a_n|$ diverges, we cannot deduce that $\sum a_n$ diverges.

PROOF:

If $b_n = a_n + |a_n|$, then $b_n \leq 2|a_n|$. Since $\sum|a_n|$ converges, it follows from the comparison test that $\sum b_n$ must converge. Furthermore, $\sum\big(b_n - |a_n|\big)$ converges, since both $\sum b_n$ and $\sum|a_n|$ converge. Therefore, $\sum a_n$ converges, since $\sum a_n = \sum\big(b_n - |a_n|\big)$.

EXAMPLE

The alternating series $\sum \dfrac{(-1)^{n+1}}{n^2}$ is absolutely convergent, since

$$\sum\left|\frac{(-1)^{n+1}}{n^2}\right| = \sum\frac{1}{n^2}$$

is a convergent p-series (with $p = 2 > 1$).

EXAMPLE

Test for convergence of $\sum \frac{(-1)^{n+1}}{n^2+1}$.

SOLUTION

The series

$$\begin{aligned}\sum |a_n| &= \sum \left|\frac{(-1)^{n+1}}{n^2+1}\right| \\ &= \sum \frac{1}{n^2+1}\end{aligned}$$

is absolutely convergent by the comparison test with the test series $\sum \frac{1}{n^2}$ (p-series with $p = 2 > 1$), since

$$\frac{1}{n^2} > \frac{1}{n^2+1} \quad \text{for } n \geq 1.$$

Therefore, this series converges by the absolute convergence theorem.

EXAMPLE

Show that $\sum \frac{1+2\sin(n)}{n^2}$ is convergent.

SOLUTION

Consider the series:

$$\sum \left|\frac{1+2\sin(n)}{n^2}\right| = \sum \frac{|1+2\sin(n)|}{n^2} \quad \text{(positive term series).}$$

Noting that

$$\left|1+2\sin(n)\right| \leq 3 \quad \text{since } |\sin(n)| \leq 1,$$

then

$$\frac{\left|1+2\sin(n)\right|}{n^2} \leq \frac{3}{n^2}.$$

But, the series

$$\sum \frac{3}{n^2} = 3\sum \frac{1}{n^2}$$

is a convergent p-series ($p = 2 > 1$), and by the comparison test $\sum \frac{|1+2\sin(n)|}{n^2}$ also converges. This implies that $\sum \left|\frac{1+2\sin(n)}{n^2}\right|$ is absolutely convergent. Hence, $\sum \frac{1+2\sin(n)}{n^2}$ converges by the absolute convergence theorem.

4.4 Conditionally Convergent Theorem

A series $\sum a_n$ is said to be *conditionally convergent* if $\sum |a_n|$ diverges and $\sum a_n$ converges.

EXAMPLE

Consider the alternating harmonic series, $\sum \frac{(-1)^{n+1}}{n}$.

Let $a_n = \frac{1}{n}$. Then, $a_{n+1} = \frac{1}{n+1}$, and

$$\frac{1}{n+1} < \frac{1}{n} \quad \text{since } n+1 > n \text{ for } n > 0.$$

Hence $a_{n+1} < a_n$ for $n > 0$. Also,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Therefore, the series converges by the alternating series test. But,

$$\sum |a_n| = \sum \left| \frac{(-1)^{n+1}}{n} \right| = \sum \frac{1}{n}$$

is a divergent harmonic series. Hence, this series conditionally converges.

5 Review Questions

[1] If S_n denotes the nth partial sum of $\sum_{m=1}^{\infty} \frac{3}{(3m-2)(3m+1)}$, show that

$$S_n = 1 - \frac{1}{3n+1}.$$

Deduce that the series converges. What is its sum?

[2] If S_n denotes the nth partial sum of $\sum_{m=1}^{\infty} \frac{3m-2}{m(m+1)(m+2)}$, show that

$$S_n = 1 + \frac{1}{n+1} - \frac{4}{n+2}.$$

Deduce that the series converges. What is its sum?

[3] Determine the nth partial sum of each of the following series, and hence determine whether the series converges:

(a) $\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m(m+2)}$;

(b) $\sum_{n=0}^{\infty} (3/4)^{n+1}$;

(c) $2 - 4/3 + 8/9 - 16/27 + 32/81 + \cdots - \cdots$.

[4] For the series $\sum_{m=1}^{\infty} \log \frac{(m+1)^2}{m(m+2)}$, show that the nth partial sum is

$$S_n = \log 2 + \log \frac{n+1}{n+2}.$$

Deduce that the series converges. What is its sum?

[5] Use the comparison test to determine which of the following series converge:

(a) $\sum \frac{n}{100n^2+1}$;

(b) $\sum \frac{1}{n\sqrt{n^3+1}}$;

(c) $\sum \frac{m}{(m+2)(m+1)}$;

(d) $\sum \frac{k+2}{\sqrt{k}\,(k+1)^2}$;

(e) $\sum \frac{2k+3}{k^2+5}$.

[6] Determine which of the following series converge:

(a) $\sum \frac{n^2}{2^n}$;

(b) $\sum \frac{3^n}{n^2}$;

(c) $\sum \frac{1}{5^n}$;

(d) $\sum \frac{2^n}{n!}$;

(e) $\sum \frac{n\,10^n}{(n+1)!}$;

(f) $\sum \frac{n^3}{5^n}$;

(g) $\sum \frac{(n!)^2}{(2n)!}$.

[7] Determine which of the following series converge:

(a) $\sum (-1)^n \frac{1}{\sqrt{n}}$;

(b) $\sum (-1)^n \frac{n}{3n+1}$;

(c) $\sum \frac{(-1)^n}{n^2+1}$;

(d) $\sum \frac{(-1)^m}{\sqrt{m^3+1}}$.

[8] Show that the series

$$\sum \frac{\sin(n\theta)}{n^2} \quad \text{and} \quad \sum \frac{\cos(n\theta)}{n^2}$$

are absolutely convergent for all θ.

[9] Determine, with reasons, which of the following series converge:

(a) $\sum \frac{1}{n(n+1)}$;

(b) $\sum (-1)^n \frac{3^{n-2}}{4^{n+1}}$;

(c) $\sum \frac{2(-1)^n}{\sqrt{4n+1}}$;

(d) $\sum \frac{4m^2}{2^m}$;

(e) $\sum \frac{1}{\sqrt{4n^3+\log n}}$;

(f) $\sum \frac{e^n}{n!}$

(g) $\sum \left(\frac{5}{6}\right)^n \left(\frac{n+2}{n+1}\right)$;

(h) $\sum \frac{(m+2)!}{m!\, m^2}$;

(i) $\sum \frac{3^k k!}{k^k}$;

(j) $\sum \frac{m}{2^{m^2}}$.

6 Answers to Review Questions

[1] $\lim_{n\to\infty} S_n = 1$

[2] $\lim_{n\to\infty} S_n = 1$

[3] (a) $S_n = \frac{1}{2}\Big[\frac{1}{2} + (-1)^n\big(\frac{1}{n+2} - \frac{1}{n+1}\big)\Big]$; series converges with a sum of $1/4$.

(b) $S_n = 3\big(1 - (3/4)^n\big)$; series converges with a sum of 3.

(c) $S_n = \frac{6}{5}\big(1 - (-2/3)^n\big)$; series converges with a sum of $6/5$.

[4] Series converges with a sum of $\log 2$.

[5] (a) Divergent

(b) Convergent

(c) Divergent

(d) Convergent

(e) Divergent

[6] (a) Convergent

(b) Divergent

(c) Convergent

(d) Convergent

(e) Convergent

(f) Convergent

(g) Convergent

[7] (a) Convergent

(b) Divergent

(c) Convergent

(d) Convergent

[8] Not available.

[9] (a) Convergent; comparison test.

(b) Convergent geometric series (common ratio, $r = -3/4$).

(c) Convegent; alternating series test.

(d) Convergent; ratio test.

(e) Convergent; comparison test.

(f) Convergent; ratio test.

(g) Convergent; ratio test.

(h) Divergent; test for a divergent series.

(i) Divergent; ratio test.

(j) Convergent; ratio test.